NDU

MAT 335

Partial Differential Equations

Exam# 1

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Duration: 60 minutes

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Section: MWF 9 → 10

Grade: 94

1) **(20%)** The solution of the insulated $1-d$ heat flow problem:

PDE: $u_t = \alpha^2 u_{xx}$ $\quad 0 < x < 1, \quad 0 < t < \infty$

BCs: $u(0,t) = 0$ $\quad 0 < t < \infty$

$u(1,t) = 0$ $\quad 0 < t < \infty$

IC: $u(x,0) = \phi(x)$ $\quad 0 \le x \le 1$

is given by $u(x,t) = \sum_1^\infty A_n e^{-(n\pi\alpha)^2 t} . \sin n\pi x$, where $A_n = 2\int_0^1 \phi(x) \sin n\pi x dx$.

Find the solution of the following $1-d$ heat flow problem with lateral heat loss:

PDE: $u_t = \alpha^2 u_{xx} - \beta u$ $\quad 0 < x < 1, \quad 0 < t < \infty$

BCs: $u(0,t) = 0$ $\quad 0 < t < \infty$

$u(1,t) = 0$ $\quad 0 < t < \infty$

IC: $u(x,0) = \psi(x)$ $\quad 0 \le x \le 1$

Hint: Consider the transformation $u(x,t) = e^{-\beta t} . w(x,t)$ to rewrite IBVP in terms of w, then use (a). (α, β are positive constants)

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$u(x,t) = e^{-\beta t} . w(x,t)$

PDE $\Rightarrow$ $\cancel{-\beta e^{-\beta t} . w(x,t)} + \cancel{e^{-\beta t}} w_t(x,t) = \alpha^2 \cancel{e^{-\beta t}} w_{xx}(x,t) \cancel{- \beta e^{-\beta t} w(x,t)}$

$\boxed{w_t(x,t) = \alpha^2 w_{xx}(x,t)}$ PDE

$u(0,t) = 0 \Rightarrow e^{-\beta t} . w(0,t) = 0 \Rightarrow \boxed{w(0,t) = 0}$ BC1

$u(1,t) = 0 \Rightarrow e^{-\beta t} . w(1,t) = 0 \Rightarrow \boxed{w(1,t) = 0}$ BC2

$u(x,0) = \psi(x) \Rightarrow 1 . w(x,0) = \psi(x) \Rightarrow \boxed{w(x,0) = \psi(x)}$ IC

$\boxed{w(x,t) = \sum_1^\infty A_n e^{-(n\pi\alpha)^2 t} \sin n\pi x}$

2) **(30%)** Solve the following <u>nonhomogeneous</u> IBVP:

PDE: $u_t = \alpha^2 u_{xx} + \sin 3\pi x \qquad 0 < x < 1, \qquad 0 < t < \infty$

BCs: $u(0,t) = 0 \qquad 0 < t < \infty$

$u(1,t) = 0 \qquad 0 < t < \infty$

IC: $u(x,0) = \sin \pi x \qquad 0 \le x \le 1$

Show all details.

$$u(x,t) = \sum T_n(t) \sin n\pi x$$

$$T_n'(t) \sin n\pi x = \alpha^2 (-n\pi)^2 \underset{T_n(t)}{} \sin n\pi x + \sin 3\pi x$$

$$\left(T_n'(t) + (\alpha n\pi)^2 T_n(t)\right) \sin n\pi x = \sin 3\pi x$$

for $n = 3$ $\quad T_3'(t) + (3\alpha\pi)^2 T_3(t) = 1$

$n \neq 3$ $\quad T_n'(t) + (n\alpha\pi)^2 T_n(t) = 0$

1st order differential equation

IC: $u(x,0) = \sin \pi x$

$$\left(T_m'(0) + (\alpha m\pi)^2 T_m(0)\right) \sin m\pi x = \sin \pi x$$

for $m = 1$ $\quad T_1'(0) + (\alpha\pi)^2 T_1(0) = 1$

$m \neq 1$ $\quad T_m'(0) + (\alpha m\pi)^2 T_m(0) = 0$

$\underline{m = 3}$ $\quad T_3'(t) + (3\alpha\pi)^2 T_3(t) = 1 \implies T_3(t) = C e^{-(3\alpha\pi)^2 t} + \frac{1}{(3\alpha\pi)^2}$

$T_3(0) = 0$

$\iff C = -\frac{1}{(3\alpha\pi)^2}$

$$\implies T_3(t) = \frac{1}{(3\alpha\pi)^2}\left(1 - e^{-(3\alpha\pi)^2 t}\right)$$

$\underline{n=1}$ $\quad T_1'(t) + (\alpha\pi)^2 T_1(t) = 0$

$$\Rightarrow T_1(t) = c\, e^{-(\alpha\pi)^2 t}$$

$$T_1(0) = 1$$

$$\Rightarrow \quad c = 1$$

$$\Rightarrow T_1(t) = e^{-(\alpha\pi^2)t}$$

$$\begin{aligned} u(x,t) &= \sum T_n(t) \sin n\pi x \\ &= \sum e^{-(\alpha\pi^2)t} \sin \pi x + \frac{1}{(3\alpha\pi)^2}\left(1 - e^{-(3\alpha\pi)^2 t}\right) \sin 3\pi x \end{aligned}$$

3) (25%)

a) Use the Fourier transform to solve the IVP:

PDE: $u_t = \alpha^2 u_{xx} - \beta u \qquad -\infty < x < \infty,\ \ 0 < t < \infty$

IC: $u(x,0) = \phi(x) \qquad -\infty < x < \infty$

Show all details.

b) What is the solution in the special case $\phi(x) \equiv 1$?

a)

Let $U(\xi, t) = \mathcal{F}(u(x,t))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\xi x}\, dx$

PDE

$\Rightarrow U_t(\xi,t) = -\alpha^2 \xi^2 U(\xi,t) - \beta U(\xi,t)$

$U_t(\xi,t) + (\alpha^2 \xi^2 \beta) U(\xi,t) = 0 \qquad$ ODE

$U_t(\xi,t) = C\, e^{-(\alpha^2\xi^2\beta)t}$

IC: $U(\xi,0) = C = \Phi(t) \qquad \left(\Phi = \mathcal{F}\,\phi(x)\right)$

$U(\xi,t) = \Phi(t) \cdot e^{-(\alpha^2 t)\xi^2} \cdot e^{-\beta t}$

$u(x,t) = \mathcal{F}^{-1}\left(U(\xi,t)(x)\right) = \mathcal{F}^{-1}\Phi(t) * \mathcal{F}^{-1}\left(e^{-(\alpha^2 t)\xi^2}\right)$

$= e^{-\beta t}\phi(x) * \mathcal{F}^{-1}\left(e^{-(\alpha^2 t)\xi^2}\right)$

$\mathcal{F}^{-1}\left(e^{-(\alpha^2 t)\xi^2}\right) = \mathcal{F}^{-1}\left(e^{-\xi^2/4a^2}\right) = e^{-a^2x^2} \cdot a\sqrt{2}$

$\frac{1}{4a^2} = \alpha^2 t \Rightarrow a = \frac{1}{2\alpha\sqrt{t}}$

$$\Rightarrow u(x,t) = e^{-\beta t}\, \phi(x) * \frac{\sqrt{2}}{2\alpha\sqrt{t}}\, e^{-\frac{x^2}{4\alpha^2 t}}$$

$$= \frac{e^{-\beta t}}{\sqrt{2t}\,\alpha} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x-z)\, e^{-\frac{z^2}{4\alpha^2 t}}\, dz$$

$$\boxed{u(x,t) = \frac{e^{-\beta t}}{2\alpha\sqrt{t\pi}} \int_{-\infty}^{\infty} \phi(x-z)\, e^{-\frac{z^2}{4\alpha^2 t}}\, dz}$$

b) for $\phi(x) = 1$

$$u(x,t) = \frac{e^{-\beta t}}{2\alpha\sqrt{t\pi}} \int_{-\infty}^{\infty} 1 \cdot e^{-\frac{z^2}{4\alpha^2 t}}\, dz$$

let $u = \frac{z}{2\alpha\sqrt{t}}$

$$\Rightarrow du = \frac{1}{2\alpha\sqrt{t}}\, dz \Rightarrow dz = du\,(2\alpha\sqrt{t})$$

$$\Rightarrow u(x,t) = \frac{e^{-\beta t}\, \cancel{2\alpha\sqrt{t}}}{\cancel{2\alpha\sqrt{t\pi}}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\, du}_{\sqrt{\pi}}$$

$$= e^{-\beta t} \frac{\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}}$$

$$= \boxed{e^{-\beta t}}$$

4) **(10%)** Let $u = u(x, y, z)$ be a function of 3 independent variables possessing derivatives of all orders. Solve the (possibly easiest) 3[rd] order PDE:

$u_{xyz} = 0, \qquad x, y, z \in \mathbb{R}.$

$$u = u(x, y, z) \qquad u_{xyz} = 0$$

$$\int u_{xyz}\, dx = \int 0\, dx$$

$$u_{yz} = f(y, z)$$

$$\int u_{yz}\, dy = \int f(y, z)\, dy$$

$$u_z = f(y, z) + f(x, z)$$

$$\int u_z\, dz = \int (f(y, z) + f(x, z))\, dz$$

$$u = f_1(y, z) + f_2(x, z) + f_3(x, y)$$

where f_1, f_2, f_3 are all deferentibal equations

5) (15%) Solve the following <u>nonlinear</u> IVP:

PDE: $u_t + uu_x = 0$ $\qquad -\infty < x < \infty,\quad 0 < t < \infty$

IC: $u(x,0) = x+1$ $\qquad -\infty < x < \infty$

Hint: a) Try a solution of the form $u(x,t) = X(x)T(t)$.

b) The general solution of the 1st order nonlinear ODE

$$T' + kT^2 = 0 \quad \text{is} \quad T(t) = \frac{1}{kt+b}$$

$u(x,t) = X(x)T(t)$

$XT' + XT \cdot X'T = 0$

$\Rightarrow \cancel{X}T' = -XX'T^2$

$\Rightarrow \frac{T'}{T^2} = -X' = \alpha$

$\Rightarrow T' - \alpha T^2 = 0$ and $X' = -\alpha$

let $\alpha = -K$

$\Rightarrow T' + KT^2 = 0$ and $X' = K$ $\qquad \boxed{X = Kx + a}$

1 $\Rightarrow T(t) = \frac{1}{Kt+b}$ $\qquad$ 2

$(X) =$

$$\boxed{u(x,t) = \frac{Kx+a}{Kt+b} = \frac{x+A}{t+B}} \Rightarrow u(x,0) = \frac{x+A}{B} = x+1$$

$A = B = 1 \Rightarrow u = \frac{x+1}{t+1}$

$u(x,0) = \cancel{x+1}$

$\Rightarrow \frac{1}{b} = x+1 \Rightarrow b(x+1) = 1 \Rightarrow \boxed{b = \frac{1}{x+1}}$

$$T(t) = \frac{1}{Kt + \frac{1}{x+1}} = \frac{1}{\frac{Kt(x+1)+1}{x+1}} = \boxed{\frac{x+1}{Kt(x+1)+1}}$$

$X' = K$

$\Rightarrow \boxed{X(x) = x}$

$$u(x,t) = X(x)\,T(t)$$
$$= x\left(\frac{x+1}{kt(x+1)+1}\right) \qquad K = \text{cst}$$